into the cone K_+, therefore the matrix $U(t, s)$ $(t_0 \leqslant s \leqslant t < \infty)$ transforms the cone K_0 into itself. The Lemma is proved.

Theorem 1. Let a constant λ_0 exist under the conditions of the Lemma, such that

$$\lambda_0 > \max \Lambda_i \qquad (1 \leqslant i \leqslant n-1) \tag{9}$$

$$P(t, \lambda_0) \geqslant 0 \qquad (t_0 \leqslant t < \infty) \tag{10}$$

Then the following estimate holds:

$$|U(t, s)| \leqslant Me^{\lambda_0(t-s)} \qquad (t_0 \leqslant s \leqslant t < \infty) \tag{11}$$

where M is some number.

Proof. We set $u_0 = (1, \lambda_0, \ldots, \lambda_0^{n-1})$. It is easy to see that $Au_0 = (1, Q_2(\lambda_0), \ldots, Q_n(\lambda_0))$ (where $Q_k(\lambda)$ are the polynomials (4)). It follows that the vector u_0 will lie within the cone K_0. Therefore we can introduce the following equivalent norm (so called u_0-norm [2]) in the space R^n:

$$|u|_{u0} = \min \alpha \qquad (-\alpha u_0 \leqslant^{\circ} u \leqslant^{\circ} \alpha u_0)$$

We will analyze the function $u_0(t) = e^{\lambda_0(t-s)}$. From the inequality (10) it follows that:

$$\frac{du_0(t)^{\circ}}{dt} \geqslant Q(t)\, u_0(t) \qquad (t \geqslant s \geqslant t_0) \tag{12}$$

Let now $-u_0 \leqslant^{\circ} u \leqslant^{\circ} u_0$. Then from (12) it follows that:

$$-e^{\lambda_0(t-s)} u_0 \leqslant^{\circ} U(t, s)\, u \leqslant^{\circ} e^{\lambda_0(t-s)} u_0 \qquad (t \geqslant s \geqslant t_0)$$

and

$$|U(t, s)|_{u_0} \leqslant e^{\lambda_0(t-s)} \qquad (t \geqslant s \geqslant t_0) \tag{13}$$

Inequality (13) proves the inequality (11). The theorem is proved.

Corollary. Let $\lambda_0 < 0$ under the condition of Theorem 1, consequently solutions of (1) are exponentially stable.

Theorem 2. Let the inequality (10) under the conditions of Theorem 1 be replaced by

$$P(t, \lambda_0) \leqslant 0 \qquad (t_0 \leqslant t < \infty, \lambda_0 > 0)$$

Then the zero solution of (1) is unstable.

The proof of Theorem 2 which resembles that of Theorem 1, is omitted.

BIBLIOGRAPHY

1. Krein, M.G. and Rutman, M.A., Linear operators leaving a cone invariant in the Banach space. Usp. matem. nauk, Vol. 3, №1, 1948.
2. Krasnosel'skii, M.A., Positive Solutions of Operator Equations. M. Fizmatgiz, 1962.

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ON THE INSTABILITY OF A PLANE TANGENTIAL DISCONTINUITY

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The problem of instability of a plane tangential discontinuity which was already considered in [1, 2], is solved here in connection with the problem on reflection of plane monochromatic waves from a surface of discontinuity. Dependence of the decremental built-up of the perturbation waves on the Mach number and on the angle formed by the direction of motion of the perturbation wave and the flow velocity vector is obtained, and analyzed for the identical media.

Coefficient of reflection of the plane monochromatic waves for the surface of the tangential discontinuity, i. e. the ratio of the energy flux density components normal to the surface in the reflected and incident waves, has the form

$$R = \left| \frac{1 - Z}{1 + Z} \right|^2, \qquad Z = \frac{\rho c^2}{\rho' c'^2} \frac{\sin 2\theta'}{\sin 2\theta} \tag{1}$$

Here primed quantities refer to the medium moving with velocity v and unprimed quantities — to the medium at rest. Angle of refraction θ' (or θ) is related to the angle of incidence θ (or θ') by the expression

$$\sin\theta' = \frac{c'}{c} \frac{\sin\theta}{|1 - (u/c)\sin\theta|}, \quad u = v\cos\varphi \tag{2}$$

where φ is the angle between the velocity vector $\mathbf{v}$ and the projection $k_{\parallel}$ of the wave vector $\mathbf{k}$ on the plane of tangential discontinuity. Apparently the author of [3] was first to obtain the expression (1) correctly.

For the total reflection when $R = 1$, the quantity Z in (1) and the component $k_{\perp}'$ (or $k_{\perp}$) of the wave vector $\mathbf{k}'$ (or $\mathbf{k}$) normal to the surface, are purely imaginary. These cases have been studied repeatedly (see e. g. [3]).

We shall investigate the more interesting cases of the absence of reflection in more detail. Condition $R = 0$ and relations (1) and (2) yield the following algebraic equation of the sixth order in $\sin\theta$

$$\frac{(\rho/\rho')^2}{[1 - (u/c)\sin\theta]^4} - 1 = \frac{\rho c^2}{\rho' c'^2} \frac{1}{\sin^2\theta} \left\{ \frac{\rho/\rho'}{[1 - (u/c)\sin\theta]^2} - \frac{\rho' c'^2}{\rho c^2} \right\} \tag{3}$$

Using the relation

$$\sin\theta = k_{\parallel}/k = ck_{\parallel}/\omega \tag{4}$$

we can transform (3) into the form similar to that of Eq. (18) of [2] used there as the initial equation in determining the unstable oscillations of the surface of a tangential discontinuity.

In general, Eq. (3) has six solutions. In order to achieve a clear picture, it is expedient to obtain these solutions in their analytical form. This can be done, provided we assume that

$$\rho c^2 = \rho' c'^2 \tag{5}$$

The above equation holds, in particular, for the perfect gases with the same ratio of the specific heats $\gamma = C_p/C_V$. When the condition (5) holds, Eq. (3) clearly splits into two separate equations

$$\frac{c'^2/c^2}{[1 - (u/c)\sin\theta]^2} - 1 = 0, \qquad \frac{c'^2/c^2}{[1 - (u/c)\sin\theta]^2} - \frac{1}{\sin^2\theta} + 1 = 0 \tag{6}$$

of the second and fourth degree in $\sin\theta$, respectively.

We begin by investigating the solutions

$$\sin\theta = (c \pm c')/u \tag{7}$$

of the first equation of (6). Analyzing these solutions we find that they have a physical meaning, i. e. the waves are bounded at infinity only for $u \geqslant u_{1,2} = |c \pm c'.|$

Inserting the solutions (7) into the refraction equation (2) we find that $\sin\theta' = \sin\theta$, consequently solutions (7) correspond to the waves which pass through the tangential discontinuity from one medium to the other without reflection ($R = 0$) or refraction ($\theta' = \theta$). In the case of identical media ($\rho' = \rho$, $c' = c$) there exists only one such wave for $u \geqslant u_1 = 2c$; the other wave degenerates ($\omega \to \infty$).

Waves with normal incidence on the surface of the tangential discontinuity represent a particular case
$$Z = -\frac{\rho c}{\rho' c'}, \qquad R = \left(\frac{\rho c - \rho' c'}{\rho c + \rho' c'}\right)^2$$

If the media are identical, $R = 0$ for any u.

We now turn our attention to solutions of the second equation of (6). These are relatively simple when the media are identical
$$2 / \sin\theta = u / c \pm \sqrt{(u / c)^2 + 4 + 4 \sqrt{(u/c)^2 + 1}} \quad (8)$$
$$2 / \sin\theta = u / c \pm \sqrt{(u / c)^2 + 4 - 4 \sqrt{(u / c)^2 + 1}} \quad (9)$$

We can easily see that solutions (8) are real for any u, while solutions (9) are real only for $u \geqslant u_0 = 2\sqrt{2}c$. In both cases $|\sin\theta| \leqslant 1$. Real solutions correspond to the waves which pass from one medium to the other through the tangential discontinuity without reflection ($R = 0$), but are refracted ($\theta' \neq \theta$).

For $u < u_0$ solutions (9) for $\sin\theta$ are complex. From the relation (4) it follows that when $k_{\parallel}$ is real, ω are complex. Consequently for $u < u_0$ solutions (9) correspond to the waves one of which increases with time , while the other (whose frequency is the complex conjugate of the first one) decays. Existence of the first wave ascertains the instability of the tangential discontinuity. Components $k_{\perp}'$ and $k_{\perp}$ of the wave vector of this wave are also complex and $k_{\perp}' = k_{\perp}^*$. Consequently these waves are of the generalized-surface-wave type (this term is borrowed from the theory of the Rayleigh's waves in a solid).

Let us now see under which conditions the instability will be maximum. Using the condition $d\mathrm{Im}\omega / du = 0$ we can easily find $u = u_m$ at which the perturbation wave growth decrement reaches its maximum value. Using (9) we find $u_m = \sqrt{3c}$ and we also obtain $\mathrm{Im}\omega = ck_{\parallel} / 2$. Thus, the maximum wave growth decrement is inversely proportional to the wavelength $\lambda_{\parallel}$.

Now from (2) we have $u = v\cos\varphi$. Then for the given velocity $v < u_m$ the wave with $\varphi = 0$, i.e. the wave with $k_{\parallel}$ directed along v , will exhibit maximum instability. If $v > u_m$, the most unstable will be the wave with $\varphi = \varphi_m = \arccos(u_m / v)$ and the magnitude of the maximum instability will be the same for any $v > u_m$. When $v > u_0$, the perturbation wave will become stable for certain values of φ [2], satisfying the inequality $0 < \varphi < \varphi_0$ where $\varphi_0 = \arccos(u_0/v)$ and φ_0 will increase with increasing velocity of flow.

The analysis of the more general case for different media with the assumption (5) omitted is substantially more difficult; nevertheless, the principles noted remain qualitatively valid and only numerical values, particularly in u_0, u_m, φ_0 and φ_m, will be subjected to changes.

BIBLIOGRAPHY

1. Landau, L. D., On the stability of tangential discontinuities in a compressible fluid. Dokl. Akad. Nauk SSSR, Vol. 44, №4, 1944.

2. Syrovatskii, S. I., Instability of tangential discontinuities in a compressible medium. ZhEFT, Vol. 27, №1, 1954.

3. Miles, J. W., On the reflection of sound at an interface of relative motion. J. Acust. Soc. Amer., Vol. 29, №2, 1957.

Translated by L. K.